Cross Smoothness Parameter Estimation for Bivariate Gaussian Processes

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Background

- Gaussian process (GP) models form a class of probabilistic models widely used in spatial statistics, time series and other fields. When Gaussian process models are applied, the smoothness parameters in their covariance functions are usually unknown, and need to be estimated from data.
- Kent and Wood (1997) constructed increment-based estimators for the smoothness parameter of a class of univariate GP using regularly spaced observations on $[0, 1]$.
- Zhou and Xiao (2018) extended the methods of Kent and Wood (1997) to bivariate case $X = (X_1, X_2)$ and studied the joint asymptotic behavior of smoothness estimators for $Var(X_1)$ and $Var(X_2)$.

Model Setup

Consider a bivariate stationary Gaussian process $X = \{(X_1(t), X_2(t))^T, t \in \mathbb{R}\}$ with zero mean and covariance function

$$C(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{pmatrix}. \tag{1}$$

Assume that

$$C_{ii}(t) = \sigma_i^2 - c_{ii}|t|^{\alpha_{ii}} + o(|t|^{\alpha_{ii}}), \tag{2}$$

$$C_{ij}(t) = \rho\sigma_1\sigma_2(1 - c_{12}|t|^{\alpha_{12}} + o(|t|^{\alpha_{12}}), \tag{3}$$

where $\sigma_i, c_{ii}, c_{ij} > 0$, $\alpha_{ii} \in (0, 2)$, $|\rho| \in (0, 1)$, $i, j \in \{1, 2\}$, $i \neq j$; also $\alpha_{12} > (\alpha_{11} + \alpha_{22})/2$.

Further assume the condition (A_q) in Kent and Wood (1997) holds for the qth derivative of covariance function C_{ij}, that is,

$$C_{ij}^{(q)}(t) = -A_{ij}\frac{\alpha_{ij}!}{q!}|t|^{\alpha_{ij}-q} + o(|t|^{\alpha_{ij}-q}) \tag{4}$$

as $|t| \to 0$, where $q \geq 1$, $i, j \in \{1, 2\}$, $A_{ii} = c_{ii}$, $A_{12} = A_{21} = \rho\sigma_1\sigma_2c_{12}$, $\alpha_{ij}!/q! = \alpha_{ij}(\alpha_{ij} - 1)\ldots(\alpha_{ij} - q + 1)$.

Covariations

Let $a = (a_{-J}, \ldots, a_J)^T$ be an increment of order p. Denote $X_{n,i}^u \in \mathbb{R}^{n(2J+1)}$ the vector of observations of component X_i, where $i = 1, 2$, $u = 1, \ldots, m$ and $n \in \mathbb{Z}^+$. For $j = 1, 2, \ldots, 2J + 1$ and $k = 1, \ldots, n$,

$$(X_{n,i}^u)_{j+(k-1)(2J+1)} = X_i\left(\frac{k + u(j - J - 1)}{n}\right).$$

Define

$$Y_n^u := \begin{pmatrix} Y_{n,1}^u \\ Y_{n,2}^u \end{pmatrix} = \begin{pmatrix} n^{\alpha_{11}/2}(I_n \otimes a^T) & 0 \\ 0 & n^{\alpha_{22}/2}(I_n \otimes a^T) \end{pmatrix} X_n^u,$$

where $\otimes$ denotes the Kronecker product, $X_n^u = \begin{pmatrix} X_{n,1}^u \\ X_{n,2}^u \end{pmatrix}$. More specifically, for $k = 1, \ldots, n$,

$$Y_{n,i}^u(k) = n^{\alpha_{ii}/2}\sum_{j=1}^{2J+1} a_{j-J-1}(X_{n,i}^u)_{j+(k-1)(2J+1)}.$$

Denote $Z_{n,12}^u(j) = n^{\alpha_{12}-(\alpha_{11}+\alpha_{22})/2}Y_{n,1}^u(j)Y_{n,2}^u(j)$ for $j = 1, \ldots, n$.

The covariation is defined as

$$\bar{Z}_{n,12}^u = \frac{1}{n}\sum_{j=1}^{n} Z_{n,12}^u(j). \tag{5}$$

Theorem 1

Denote $A = -\rho\sigma_1\sigma_2c_{12}\sum_{k,l} a_k a_l|k - l|^{\alpha_{12}}$.

- For any $j = 1, \ldots, n$ and any $u = 1, \ldots, m$,
$$E[Z_{n,12}^u(j)] \to Au^{\alpha_{12}} \quad \text{as } n \to \infty,$$
where $A = 0$ if $\alpha_{12}/2 \in \mathbb{Z}$ and $p \geq \alpha_{12}/2$.
- Denote $\bar{Z}_{n,12} = (\bar{Z}_{n,12}^1, \ldots, \bar{Z}_{n,12}^m)^T$ and take $p \geq 1$. When $\alpha_{11} + \alpha_{22} < 2\alpha_{12}$ and (4) holds for $q = 2p + 2$,
$$n^{1/2+(\alpha_{11}+\alpha_{22})/2-\alpha_{12}}(\bar{Z}_{n,12} - E\bar{Z}_{n,12}) \Rightarrow N(0, \Phi) \tag{6}$$
as $n \to \infty$ for some constant matrix $\Phi \in \mathbb{R}^{m\times m}$.

Main Results

Define the estimator of α_{12} as

$$\hat{\alpha}_{12} = \frac{1}{2}\sum_{u=1}^{m} L_u \ln(\bar{Z}_{n,12}^u)^2 \tag{7}$$

$$= \frac{1}{2}\sum_{u=1}^{m} L_u \ln\left(X_n^{uT}\begin{pmatrix} 0 & I_n \otimes (aa^T) \\ I_n \otimes (aa^T) & 0 \end{pmatrix} X_n^u\right)^2, \tag{8}$$

where $\{L_u, u = 1, \ldots, m\}$ is a list of constants satisfying $\sum_{u=1}^m L_u = 0$ and $\sum_{u=1}^m L_u \ln u = 1$,

Theorem 2

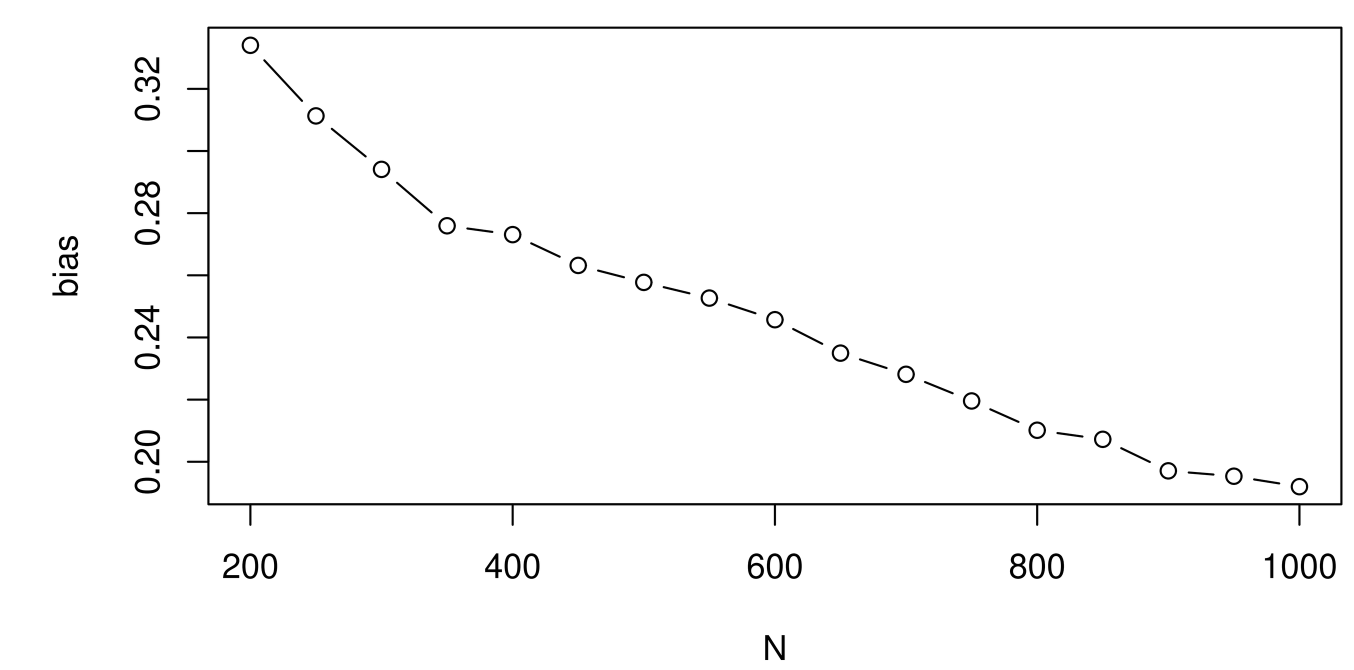
- Assume (4) holds for $q = 2p + 3$. When $\alpha_{11} + \alpha_{22} < 2\alpha_{12} < \alpha_{11} + \alpha_{22} + 1 < 4p + 4$ or $4p + 3 < \alpha_{11} + \alpha_{22} < 2\alpha_{12} < 4p + 4$,
$$\hat{\alpha}_{12} \xrightarrow{a.s.} \alpha_{12} \quad \text{as } n \to \infty$$
if $A \neq 0$.
- Take $p \geq 1$. Assume (4) holds for $q = 2p + 2$ and
$$C_{12}(t) = C_{21}(t) = \rho\sigma_1\sigma_2(1 - c_{12}|t|^{\alpha_{12}} + O(|t|^{\alpha_{12}+\beta_{12}}) \quad \text{as } t \to 0 \tag{9}$$
for some $\beta_{12} > 0$. If $A \neq 0$, $\alpha_{11} + \alpha_{22} < 2\alpha_{12}$ and $\alpha_{12} + \beta_{12} > (\alpha_{11} + \alpha_{22} + 1)/2$, then
$$n^{1/2+(\alpha_{11}+\alpha_{22})/2-\alpha_{12}}(\hat{\alpha}_{12} - \alpha_{12}) \Rightarrow N(0, A^{-2}\tilde{L}^T\Phi\tilde{L}) \tag{10}$$
as $n \to \infty$, where $\tilde{L} = (L_1, L_2/2^{\alpha_{12}}, \ldots, L_m/m^{\alpha_{12}})^T \in \mathbb{R}^m$.

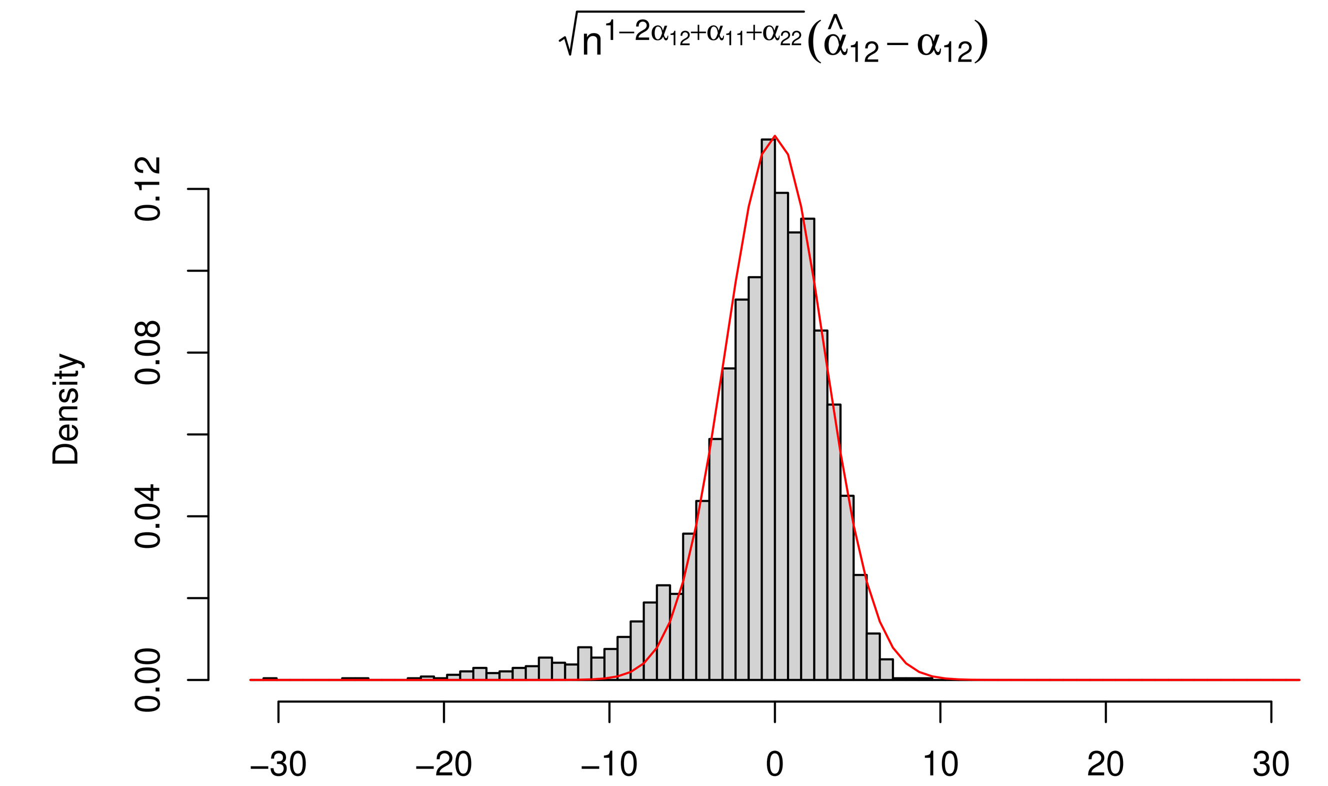
Simulation

Denote $M_\nu(t) = 2^{1-\nu}\Gamma(\nu)^{-1}|t|^\nu K_\nu(|t|)$ the Matérn covariance function. Take $C_{11} = C_{22} = M_{0.5}$ and $C_{12} = C_{21} = 0.5M_{0.55}$. Let $m = 50$, $p = 1$, $a = (1, -2, 1)^T$ and $n \in \{200, 250, \ldots, 1000\}$. For each value of n, generate 3000 independent realizations of the process X.

In this case, $\sigma_1 = \sigma_2 = 1$, $\rho = 0.5$, $\alpha_{12} = 1.1$, $c_{12} = 0.5^{1.1}\Gamma(1 - 0.55)/\Gamma(1 + 0.55)$, $A \approx 1.9177$.

Simulation results shown in the following two figures illustrate the asymptotic behavior of $\hat{\alpha}_{12}$.





Future Directions

The estimator discussed above is constructed based on observations of X on regular grids. Studying irregular sampling designs would be one direction for future research. Confidence intervals and hypothesis testings regarding α_{12} are also interesting to consider.

References

[1] John T. Kent and Andrew T. A. Wood. Estimating the fractal dimension of a locally self-similar gaussian process by using increments. *Journal of the Royal Statistical Society. Series B (Methodological)*, 59(3):679–699, 1997.

[2] Yuzhen Zhou and Yimin Xiao. Joint asymptotics for estimating the fractal indices of bivariate gaussian processes. *Journal of Multivariate Analysis*, 165:56–72, 2018.

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